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1982 J. Phys. A: Math. Gen. 15 L357

(http://iopscience.iop.org/0305-4470/15/7/006)

J. Phys. A: Math. Gen. 15 (1982) L357–L361. Printed in Great Britain

LETTER TO THE EDITOR

$Z(N)$ generalisation of the Baxter–Wu model

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Received 22 February 1982

Abstract. A two-dimensional model of the dynamics of $Z(N)$ spins on a triangular lattice with three-body interactions is analysed. The properties of the model under duality are shown to be identical to those of a wide class of models in two, three and four dimensions. Monte Carlo simulations reveal the existence of three phases for $N>4$, with an intermediate phase which is most likely soft.

A profound and, as yet, unclear relationship seems to exist between a class of spin systems in two and three dimensions with global or less than global $Z(N)$ symmetry and gauge theories in four dimensions with a local $Z(N)$ symmetry. At the root of this relationship seems to be the response of these models to a duality transformation. With this in mind, we have constructed a two-dimensional model of $Z(N)$ spins which has the same duality properties as the two-dimensional clock model on a square lattice (Elitzur *et al* 1979, Horn *et al* 1979, Ukawa *et al* 1980, Cardy 1980, Alcaraz and Köberle 1981), a four-spin model defined on a face-centred cubic lattice (Alcaraz *et al* 1981, 1982b) and a four-dimensional lattice gauge theory (Elitzur *et al* 1979, Horn *et al* 1979, Creutz *et al* 1979a, b, Ukawa *et al* 1980) with Wilson's action (Wilson 1974). The model turns out to be a generalisation of the solvable Baxter–Wu model (Baxter and Wu 1973); it is defined by spins which take the values $\exp(2\pi i n/N)$ with $n=0,1,\ldots,N-1$, on a triangular lattice with Hamiltonian

$$H=-\frac{J}{2}\sum_{\tau}(S_\tau+S_\tau^\dagger) \qquad (1)$$

where the triangle variables S_τ are defined by the product of three spins occupying the vertices of an elementary triangle and the sum is over all such simplexes. The Baxter–Wu model corresponds to $N=2$ whereas the $N\rightarrow\infty$ limit, the U(1) model, seems to be related to two-dimensional melting (Nelson 1978, Nelson and Halperin 1978, Young 1979). A detailed analysis of this connection is presently under investigation (Alcaraz *et al* 1982a).

Notice that the model defined by (1) does not have the global $Z(N)$ group as a symmetry; rather, it is invariant under $Z(N)$ transformations which act only on the spins defining an elementary hexagonal sublattice. The rich phase structure of the model is due to the possibility of spontaneously breaking these almost global symmetries involving $\frac{2}{3}$ of the spins. Using standard techniques (Savit 1980) one can show (Alcaraz and Jacobs 1982) that the Villain approximation to the partition function

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defined by (1) satisfies

$$Z_v(\beta) = cZ_v(\beta^*) \tag{2}$$

where c is a field-independent factor and $\beta^* = N^2/4\pi^2\beta^2$. That is, the theory is self-dual.

It is possible to define the most general ferromagnetic triplet model on this lattice. The Hamiltonian is given by a sum of terms of the form given in (1)

$$H = -\sum_\tau \left(\sum_{\alpha=1}^{M} \frac{J_\alpha}{2} (S_\tau^\alpha + S_\tau^{\alpha^+} - 2) \right) \tag{3}$$

where M is the integer part of $N/2$ and J_α are M positive coupling constants. The vector model studied here corresponds to the case $J_\alpha = J\delta_{\alpha,1}$. Defining the Boltzmann weights by

$$\chi_\xi = \exp \sum_{\alpha=1}^{M} J_\alpha \left[\cos\left(\frac{2\pi}{N}\alpha\xi\right) - 1 \right] \qquad \xi = 0, 1, \ldots, M, \tag{4}$$

one finds that

$$Z(\{\chi_\xi\}) \propto Z(\{\chi_\xi^*\}) \tag{5}$$

with the dual weights χ_ξ^* given by the $Z(N)$ Fourier transform of (4). Since $\chi_{N-\xi} = \chi_\xi$, it follows that

$$\chi_\xi^* = \frac{1}{N} \sum_{\alpha=0}^{N-1} \chi_\alpha \exp\left(\frac{2\pi \mathrm{i}}{N}\alpha\xi\right) \tag{6}$$

which is identical to the corresponding equation for the class of $Z(N)$ models mentioned above. Thus all conclusions drawn from duality in those models apply in our case as well. In particular, this implies that the models for $Z(2)$, $Z(3)$ and $Z(4)$ are self-dual; their self-dual points being given by $\tilde{\beta}(Z_2) = \ln(1+\sqrt{2})/2$, $\tilde{\beta}(Z_3) = 2\ln(1+\sqrt{3})/3$ and $\tilde{\beta}(Z_4) = 2\tilde{\beta}(Z_2)$. Moreover, while a single phase transition must occur at a self-dual point, multiple transitions are certainly allowed; the properties of one another being related by duality.

Details of our numerical procedure will be reported elsewhere (Alcaraz and Jacobs 1982); here we shall mention the main results which are available at present.

The Baxter–Wu model, the $N = 2$ special case of our theory, is solvable (Baxter and Wu 1973) and known to undergo a single second-order transition of a peculiar nature at the Onsager temperature (Domany and Riedel 1978, Kinzel *et al* 1981). It is believed that this model belongs to the four-state Potts universality class. This is, or course, validated by our numerical results. However, both the $Z(3)$ as well as the $Z(4)$ models undergo strong first-order transitions at their self-dual points with latent heats which are roughly 80% and 50%, respectively, of the energy of the high-temperature phase at criticality. A bifurcation seems to appear at $N = 5$ and a three-phase structure becomes evident for $N \geq 6$, in full analogy with the class of $Z(N)$ models mentioned above. Moreover, consistent with general arguments drawn from an analysis of the topological excitations of the theory (Savit 1980), for $N \geq 5$, the high-temperature transition point, β_c^{I}, becomes essentially independent of N and assumes its $N \to \infty$, U(1) value $\beta_c(\mathrm{U}(1)) \simeq 1$. Again, as expected from general arguments, the low-temperature transition point β_c^{II} scales with the inverse gap, approaching zero temperature as $1/N^2$ for large N. Thus, we find

$$\beta_c^{\mathrm{II}}(N) = \frac{\gamma}{1 - \cos(2\pi/N)} \tag{7}$$

with the scaling coefficient $\gamma \simeq 0.88$. The final picture which emerges from our analysis giving the N dependence of the critical points is shown in figure 1.

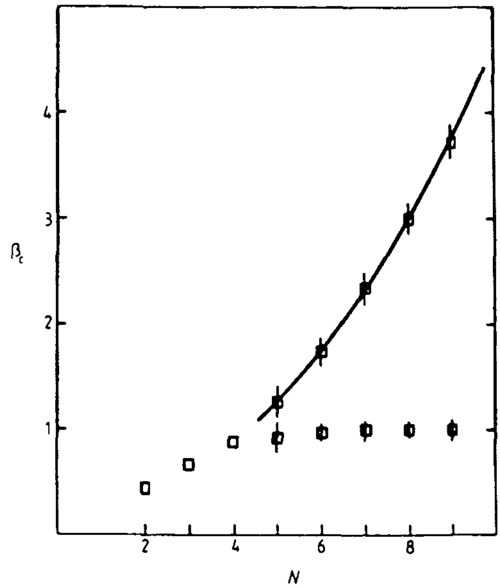

Figure 1. N dependence of the critical points. The full curve is the function $0.88/(1 - \cos 2\pi/N)$.

The intermediate phase seen for $N \geqslant 5$ is most likely of the Kosterlitz–Thouless (KT) kind (Kosterlitz and Thouless 1973), being separated from a high-temperature disordered phase and a low-temperature ordered one by infinite-order transitions. As evidence for our claim we observe the following.

The leading-order behaviour of the two-point correlation function is seen to be logarithmic (Alcaraz and Jacobs 1982) in the U(1) limit as is the case in the two-dimensional planar model (José *et al* 1977) whose low-temperature phase is KT. Even for finite N, the average energy per spin in the intermediate phase is seen to fall on the leading spin wave behaviour $E = 1/4\beta$. Consistent with this we have observed a clear signal of a transition from disorder to spin wave behaviour which is a peculiar feature of the local nature of standard Monte Carlo methods. To understand this effect (Alcaraz *et al* 1982b), consider slowly cooling a disordered initial state from $\beta = 0$ through the transition point at $\beta \simeq 1$ by slightly changing β after each Monte Carlo iteration of the entire lattice. In either a disordered or a spin wave phase the order parameter—the average spin—vanishes. However, the local (in β) fluctuations in this parameter as measured by Monte Carlo should be very different in the two cases. This is because, although in either case the spins assume all allowed values, in a spin wave phase the immediate neighbours of the spin being tested differ only slightly from it on the average. In a disordered phase, on the other hand, the difference in angle between the spin in question and its nearest neighbours is arbitrarily large. Since the updating procedure is only sensitive to local differences, this phenomenon will show itself as a quench in the spin fluctuations over small intervals of β as the system cools to $\beta \simeq 1$. That this effect indeed occurs in our theory is seen dramatically in figure 2.

Lastly, note that the vanishing of the order parameter in this phase implies the vanishing of the disorder parameter as well, since the two are related by duality. This further strongly indicates (Alcaraz and Köberle 1981) the existence of a massless excitation in this phase.

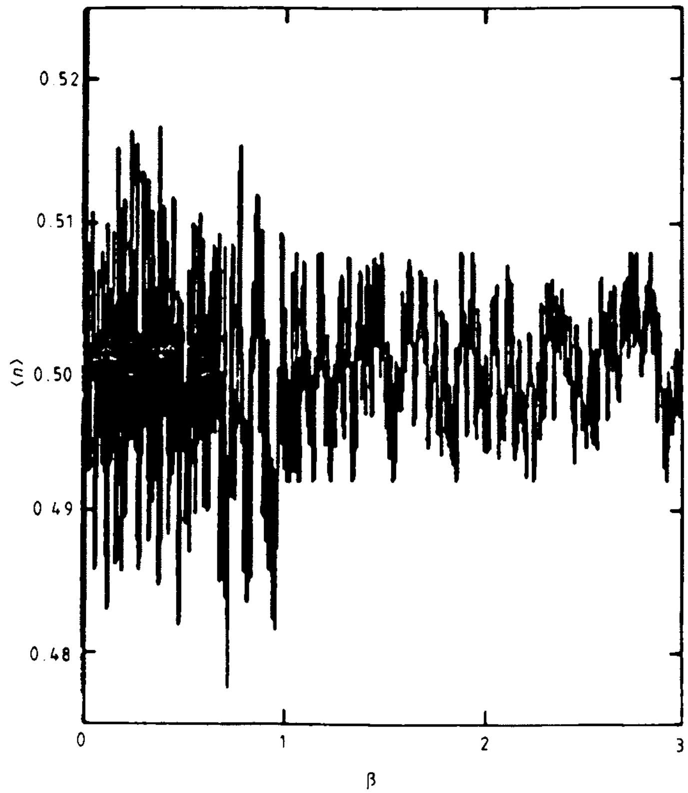

Figure 2. Quench in the fluctuations of the order parameter as the system is cooled through the high-temperature transition at $\beta \simeq 1$.

We are happy to acknowledge numerous and fruitful discussions with J Cardy, S Ostlund and R Savit. The research was supported in part through the National Science Foundation through Grant PHY77-27084.

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